in 1914, showing a decidedly lower transmission of radiation through the water cell, in the case of Venus and Saturn.

The intensity of the planetary radiation increases with decrease in the density of the surrounding atmosphere and (as interpreted from the water cell transmissions) in per cent of the total radiation emitted, is as follows: Jupiter (O), Venus (5), Saturn (15), Mars (30) and the Moon (80).

The water cell transmission of the radiations from the southern (50.6%) and northern (53.1%) hemispheres of Mars should be and is higher than that of the radiations emanating from the equatorial (47.3%) region, owing to the depletion of the reradiated energy by the greater air mass. Moreover, the intensity of the planetary radiation from the northern hemisphere of Mars was found to be less than from the southern hemisphere. This is to be expected in view of the observed cloudiness over the northern hemisphere which is usually the brighter and is approaching the winter season, and hence is at a lower superficial temperature.

These data were obtained through the generosity of the Lowell Observatory, Flagstaff, Arizona, who financed this research. Dr. C. O. Lampland again kindly operated the telescope and it is intended to publish the complete results in a joint paper on the measurements of planetary radiation and their astronomical significance.

As already stated, the object of the present note is primarily to place on record a verification of the results, obtained a year ago, on stellar temperatures.

[1] *Proc. Nat. Acad. Sci.*, **8**, 1922 (49–53).

[2] *Bureau of Standards Scientific Paper*, No. **438**, 1922.

PROOF OF A THEOREM DUE TO HEAVISIDE[1]

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In the *Electrical Papers*, Vol. 2, p. 412, Heaviside states: "The whole work done by impressed forces suddenly started exceeds the amount representing the waste by Joule-heating at the final rate (when there is any), supposed to start at once, by twice the excess of the electric over the magnetic energy of the steady field set up."

Consider a system of bodies, either conductors or dielectrics, or, for the sake of generality, both at the same time, and denote by $\mathbf{E}$ the electric force, by $\mathbf{H}$ the magnetic force, by $\mathbf{D}$ the dielectric displacement, by $\mathbf{C}$

the current of conduction, by $\mathbf{B}$ the magnetic induction. We shall suppose the matter to be isotropic, so that at each point we have to introduce only one ϵ (dielectric constant), one permeability μ, and one conductivity σ.[2] These quantities may change from place to place; they, as well as $\mathbf{E}$, $\mathbf{H}$, etc., are considered as continuous functions. If there are different bodies, transition layers in which the properties gradually change from one body to the other, are supposed to exist at the common boundary. (This is only for the purpose of mathematical convenience; the thickness of the transition layer can be supposed to diminish indefinitely, and so we can pass to the limiting case of a sharp demarcation.) Let there be two kinds of impressed forces ("electromotive" forces), the one $\mathbf{F}_1$ producing dielectric displacement, the other $\mathbf{F}_2$ producing conduction currents.[3] The meaning of this is that the dielectric displacement is determined by

$$\mathbf{D} = \epsilon(\mathbf{E} + \mathbf{F}_1) \tag{1}$$

and the conduction current by

$$\mathbf{C} = \sigma(\mathbf{E} + \mathbf{F}_2) \tag{2}$$

The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are again continuous functions of the coördinates. Each of them may be confined to a limited space and these two spaces may lie one outside the other or they may overlap more or less. At a definite point there is but one $\mathbf{E}$, the same in (1) and (2).

In addition to (1) and (2) we have

$$\mathbf{B} = \mu\mathbf{H} \tag{3}$$

$$\mathrm{curl}\,\mathbf{H} = \frac{1}{c}(\dot{\mathbf{D}} + \mathbf{C}) \tag{4}$$

$$\mathrm{curl}\,\mathbf{E} \quad - \frac{1}{c}\dot{\mathbf{B}} \tag{5}$$

$$\mathrm{div}\,(\dot{\mathbf{D}} + \mathbf{C}) = 0 \tag{6}$$

$$\mathrm{div}\,\mathbf{B} = 0 \tag{7}$$

Electric energy per unit of volume $\mathbf{D}^2/2\epsilon$.

Magnetic energy per unit of volume[4] $^1/_2(\mathbf{B}.\mathbf{H})$.

Joule-heat per unit of volume and unit of time $\mathbf{C}^2/\sigma$.

Work of $\mathbf{F}_1$ per unit of volume and unit of time $(\mathbf{F}_1.\dot{\mathbf{D}})$.

Work of $\mathbf{F}_2$ per unit of volume and unit of time $(\mathbf{F}_2.\mathbf{C})$.

We shall suppose that the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are started during an infinitely short interval of time Δt, ending at the instant $t = 0$; from this latter onward they remain constant. Finally there will be a steady state. In what follows the integrations with respect to the time are extended up to

an instant t at which the final state has been practically reached. The final values of $\mathbf{D}$, $\mathbf{C}$, etc. are denoted by $\overline{\mathbf{D}}$, $\overline{\mathbf{C}}$, etc.

The main point in the following reasoning is that, when the interval Δt is infinitely short, the values of $\mathbf{D}$ and $\mathbf{B}$ at the instant $t = 0$ are likewise infinitely small, so that we may reckon with $\mathbf{D} = 0$, $\mathbf{B} = 0$ for $t = 0$; moreover, at $t = 0$ the current $\mathbf{C}$ cannot be infinite and we may neglect the work of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ during the interval Δt. In the time-integral representing the work (up to the time t), we may take $t = 0$ for the lower limit.

Proof.—During the interval Δt the finite forces $\mathbf{F}_1$ and $\mathbf{F}_2$ cannot produce an infinite $\mathbf{E}$. Therefore, according to (5) $\dot{\mathbf{B}}$ can at the best have a finite value. But then (because at the beginning of Δt, $\mathbf{B} = 0$) $\mathbf{B}$ must be infinitely small until $t = 0$. The same is true of $\mathbf{H}$ (because connected with $\mathbf{B}$ by (3)). Therefore, on account of (4), $\dot{\mathbf{D}} + \mathbf{C}$ must be infinitely small. But, as (2) shows, $\mathbf{C}$ cannot be infinite; therefore $\dot{\mathbf{D}}$ also cannot be infinite, and $\mathbf{D}$ must be infinitely small at $t = 0$.[5] We can conclude now that both the integrals $\int(\mathbf{F}_1.\dot{\mathbf{D}})dt$ and $\int(\mathbf{F}_2.\mathbf{C})dt$ which represent the work of the electromotive forces, when taken over the infinitely small time Δt, have infinitely small values; indeed, neither $(\mathbf{F}_1.\dot{\mathbf{D}})$ nor $(\mathbf{F}_2.\mathbf{C})$ becomes infinite in the interval.

dS is the element of volume. Integrations with respect to dS are extended over the whole field, when necessary to infinite distance.

The electric energy in the final state is $U = {}^1/_2 \int \frac{\overline{\mathbf{D}}^2}{\epsilon}\, d\,S$, or, on account of (1)[6]

$$U = \frac{1}{2}\int(\{\overline{\mathbf{E}} + \mathbf{F}_1\}.\overline{\mathbf{D}})dS \tag{8}$$

The magnetic energy in the final state is $T = \frac{1}{2}\,(\overline{\mathbf{H}}.\overline{\mathbf{B}})dS$.

The Joule heat per unit of time in the final state is

$$\int\frac{\overline{\mathbf{C}}^2}{\sigma}\,dS,$$

for which we may also write[7] $\int(\mathbf{F}_2.\overline{\mathbf{C}})dS$.

If the Joule generation of heat started suddenly at $t = 0$ at the rate given by the last expression, the amount of heat developed until the time t would be[8]

$$W = \int_0^t dt \int(\mathbf{F}_2.\overline{\mathbf{C}})dS \tag{9}$$

The work of the impressed forces up to the time t is given by

$$A = \int_0^t dt \int (\mathbf{F}_1.\dot{\mathbf{D}})dS + \int_0^t dt \int(\mathbf{F}_2.\mathbf{C})dS, \tag{10}$$

where we may write for the first term $\int(\mathbf{F}_1.\overline{\mathbf{D}})dS$ because $\mathbf{F}_1$ is constant and $\int_0^t \dot{\mathbf{D}}dt = \overline{\mathbf{D}}$($\mathbf{D}$ being 0 for $t = 0$).

From (10) and (9):

$$A - W = \int(\mathbf{F}_1.\overline{\mathbf{D}})dS + \int_0^t dt\int(\mathbf{F}_2.\{\mathbf{C} - \overline{\mathbf{C}}\})dS \qquad (11)$$

In virtue of (8):

$$\int(\mathbf{F}_1.\overline{\mathbf{D}})dS = 2U - \int(\overline{\mathbf{E}}.\overline{\mathbf{D}})dS,$$

and by (2), applied to the final state, $\mathbf{F}_2 = \overline{\mathbf{C}}/\sigma - \overline{\mathbf{E}}$. Substituting these values in (11) we find

$$A - W = 2U + \int_0^t dt\int\left(\frac{1}{\sigma}\overline{\mathbf{C}}.\{\mathbf{C} - \overline{\mathbf{C}}\}\right)dS -$$
$$- \int(\overline{\mathbf{E}}.\overline{\mathbf{D}})dS - \int_0^t dt\int(\overline{\mathbf{E}}.\{\mathbf{C} - \overline{\mathbf{C}}\})dS \qquad (12)$$

Now, we may write

$$\overline{\mathbf{D}} = \int_0^t \dot{\mathbf{D}}dt,$$

and we can therefore combine the two last terms of (12) into

$$- \int_0^t dt \int(\overline{\mathbf{E}}.\{\dot{\mathbf{D}} + \mathbf{C} - \overline{\mathbf{C}}\})dS \qquad (13)$$

We have by (5) and (6)

$$\text{curl}\,\overline{\mathbf{E}} = 0, \qquad \text{div}\,(\dot{\mathbf{D}} + \mathbf{C}) = 0, \qquad \text{div}\,\overline{\mathbf{C}} = 0;$$

hence (13) vanishes[9] and (12) reduces to

$$A - W = 2U + \int_0^t dt\int\left(\frac{1}{\sigma}\overline{\mathbf{C}}.\{\mathbf{C} - \overline{\mathbf{C}}\}\right)dS \qquad (14)$$

Transformation of the last term: Substitute $\overline{\mathbf{C}} = c$ curl $\overline{\mathbf{H}}$ (form. (4) applied to final state), $\mathbf{C} - \overline{\mathbf{C}} = \sigma(\mathbf{E} - \overline{\mathbf{E}})$ (form. (2) combined with corresponding equation for final state), integrate by parts, and finally replace curl $\mathbf{E}$ by $-\dot{\mathbf{B}}/c$ and curl $\overline{\mathbf{E}}$ by 0.

$$\int_0^t dt\int\left(\frac{1}{\sigma}\overline{\mathbf{C}}.\{\mathbf{C} - \overline{\mathbf{C}}\}\right)dS = \int_0^t dt\int(c\,\text{curl}\,\overline{\mathbf{H}}.\{\mathbf{E} - \overline{\mathbf{E}}\})\,dS^{10}$$
$$= \int_0^t dt\int(c\overline{\mathbf{H}}.\text{curl}\,\{\mathbf{E} - \overline{\mathbf{E}}\})dS = -\int_0^t dt\int(\overline{\mathbf{H}}.\dot{\mathbf{B}})dS$$
$$= -\int dS\int_0^t(\overline{\mathbf{H}}.\dot{\mathbf{B}})dt \qquad (15)$$

Now, since $\overline{\mathbf{H}}$ does not depend on t, and $\mathbf{B} = 0$ for $t = 0$,

$$\int_0^t (\overline{\mathbf{H}}.\dot{\mathbf{B}})dt = (\overline{\mathbf{H}}.\overline{\mathbf{B}});$$

(15) therefore becomes

$$-\int (\overline{\mathbf{H}}.\overline{\mathbf{B}})dS = -2\,T$$

and (14)

$$A - W = 2(U - T) \qquad \text{q. e. d.} \tag{16}$$

Remarks.—1. Both A and W depend on the choice of the time t the end of the (long) interval of time which we considered. But the *difference* $A - W$ is independent of that choice. Indeed, in the steady state, the work of the impressed force per unit of time is equal to the generation of heat, equally per unit of time, so that, when the interval of time is lengthened, A and W increase by equal amounts.

When, in the final state, there are no currents (but only electric charges) $W = 0$. In this case A is independent of the choice of t.

2. W is not the quantity of heat that is *really* generated. This latter quantity W' is given by

$$W' = \int_0^t dt \int \frac{\mathbf{C}^2}{\sigma}\, dS.$$

3. In the interval of time during which the steady state is reached there will in general be a *radiation* of energy from the system outward. Let the total amount of the radiated energy (to be calculated by applying Poynting's rule to the flow through an infinitely great sphere) be R. Then, by the law of conservation of energy

$$A = W' + U + T + R.$$

Comparing this with (16) one finds

$$W - W' = 3T - U + R.$$

4. The theorem expressed by (16) can easily be verified for simple cases, when all quantities involved can be completely calculated.

(a). A linear circuit with resistance r and self-inductance L, no capacity. Let the electromotive force $\mathbf{E}$ be suddenly applied at $t = 0$, remaining constant ever afterwards

The current is given by $i = E/r.(1 - e^{-rt/L})$, and one finds

$$A = \int_0^t Eidt = \frac{E^2}{r}\,t - \frac{E^2L}{r^2} \quad (t \text{ very great}),$$

$$W = \frac{E^2}{r}\,t, \quad T = \frac{LE^2}{2r^2}, \quad U = 0.$$

(*b*). A dielectric with the constant ϵ fills all space. At time $t = 0$ an electromotive force $\mathbf{F}_1$ is suddenly applied. It has the same direction and magnitude at all points inside a sphere and is limited to that part of space. The steady state is now easily calculated; likewise the amount of energy R, that is radiated.

5. The theorem mentioned in G. W. Pierce, *Electrical Oscillations*, p. 40, is closely connected with Heaviside's general theorem.

Consider a circuit containing a condenser and in series with it a self-induction and resistance. In this case $T = 0$, $W = 0$, so that (16) becomes $A = 2U$. But, by the law of energy ($R = 0$), $A = W' + U$. Therefore $W' = U$, which is Pierce's statement.

[1] This general theorem stated without proof by Heaviside, was useful to Mr. Charles Manneback in his thesis for the doctorate in science at the Massachusetts Institute of Technology, June 1922: *Radiation from Transmission Lines* (as yet unpublished). As Dr. Manneback could not locate, in the textbook or periodical literature, a satisfactory proof of the theorem in all its generality, he appealed for help to Professor Lorentz who kindly sent the demonstration. It has seemed as though the treatment of the theorem by Dr. Lorentz with his comments upon it might be of wide interest, particularly on account of possible practical applications, and he has assented to its publication.—EDITOR.

[2] One can just as well (somewhat greater complication of the formulae) consider crystalline substances.

[3] One could add a "magneto-motive" force, but for the sake of simplicity we shall not do so.

[4] $(\mathbf{A}.\mathbf{B})$ is the scalar product of the vectors $\mathbf{A}$ and $\mathbf{B}$; $(\mathbf{A}.[\mathbf{B} + \mathbf{C}])$ the scalar product of the vectors $\mathbf{A}$ and $\mathbf{B} + \mathbf{C}$. Of course when two vectors have the same direction (as $\mathbf{B}$ and $\mathbf{H}$) we can as well take the product of their magnitudes.

[5] What is said here of $\mathbf{D}$ does not apply to the electric force. When, during the interval Δt, $\mathbf{D}$ is infinitely small (practically 0), *eq.* (1) shows that $\mathbf{E}$ is nearly $-\mathbf{F}_1$, a finite value. Note that we have drawn our conclusions from the fact that the equations contain $\dot{\mathbf{B}}$ and $\dot{\mathbf{D}}$. They do not contain $\dot{\mathbf{E}}$.

[6] We need not write $\overline{\mathbf{F}}_1$ and $\overline{\mathbf{F}}_2$, because $\mathbf{F}_1$ and $\mathbf{F}_2$ are constant from $t = 0$ onward.

[7] This formula expresses the fact that in the final state, when both U and T remain constant, the Joule heat is equal to the work of $\mathbf{F}_2$. (The work of $\mathbf{F}_1$ is 0, because in the final state $\dot{\mathbf{D}} = 0$.) One can deduce one form from the other by using $\overline{\mathbf{C}} = \sigma(\overline{\mathbf{E}} + \mathbf{F}_2)$. $\int(\overline{\mathbf{E}}.\overline{\mathbf{C}})dS = 0$, because div $\overline{\mathbf{C}} = 0$ and curl $\overline{\mathbf{E}} = 0$.

[8] Since $(\mathbf{F}_2.\overline{\mathbf{C}})$ is independent of t we could just as well write $t.\int(\mathbf{F}_2.\overline{\mathbf{C}})dS$, but in what follows, the form (9) is to be preferred.

[9] First extend the integration to the space within a sphere Σ around O, the radius r of which ultimately increases indefinitely. Since $\overline{\mathbf{E}}$ depends on a potential φ, we find by partial integration (putting $\dot{\mathbf{D}} + \mathrm{C} - \overline{\mathrm{C}} = \mathrm{Q}$) $\int(\overline{\mathbf{E}}.\mathrm{Q})dS = -\int(\mathrm{grad}\ \varphi.\mathrm{Q})dS = -\int\varphi\mathrm{Q}_n d\Sigma + \int\varphi\ \mathrm{div}\ \mathrm{Q}dS$. ($\mathrm{Q}_n$ normal component.) The surface integral vanishes for $r = \infty$ because (when the total charge of the system is 0) φ decreases at least as $1/r^2$, whereas $\mathbf{Q}_n$ becomes 0.

[10] There the integration by parts leads to the surface integral $c\int\{[\overline{\mathbf{H}}y_2(\mathbf{E}_z - \overline{\mathbf{E}}_z) - \overline{\mathbf{H}}_z(\mathbf{E}_y - \overline{\mathbf{E}}_y)]\cos\alpha + \ldots\}d\Sigma$(cos α ....direction constants of normal) to Σ and this vanishes for similar reasons as the surface integral of the preceding note.